NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from November Number.]

PROPOSITION XXX. *To any terminated straight AB stands at right angles (Fig. 36.) a certain unbounded straight BX. I say firstly, that the straight AY, erected perpendicularly toward the same parts upon AB, will be one intrinsic limit of all those straights, which drawn from the point A out toward the same parts have (in hypothesis of acute angle) a common perpendicular in two distinct points with the other unbounded straight BX. I say secondly that no acute angle will be the minimum of all, produced under which a straight from the aforesaid point A (in the aforesaid hypothesis) has in two distinct points a common perpendicular with BX.*

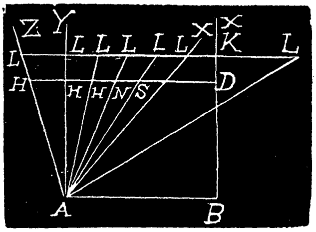

Fig. 36.

Proof of the first part.

For since AY has in common at two distinct points A and B the perpendicular AB with BX; if any straight AZ is drawn toward the same parts under an obtuse angle, it follows there can be toward these parts in two distinct points no common perpendicular to AZ, BX. Otherwise from the resulting quadrilateral containing four angles greater than four right angles, we hit (from Proposition XVI.) upon the already rejected hypothesis of obtuse angle, against the hypothesis of acute angle in this place assumed.

Therefore that perpendicular AY will be from that side an intrinsic limit of all the straights which drawn from the point A toward the same parts have (in the hypothesis of acute angle) at two distinct points a common perpendicular with the other unbounded straight BX. Quod erat primum.

Proof of the second part.

For if it were possible, let a certain acute angle be the least of all, drawn under which AN has with BX in two distinct points the common perpendicular ND. Then in BX a higher point K being assumed, from this erect to BX the perpendicular KL, upon which from the point A let fall (by Euclid I. 12) the perpendicular AL.

But now, if this AL meets ND in any point S, it certainly follows that angle BAL will be less than BAN, which therefore will not be the least of all drawn under which AN has with BX in two distinct points a common perpendicular ND.

But furthermore that the aforesaid perpendicular ND is cut by this perpendicular AL in some intermediate point of it S is thus demonstrated.

And first indeed, that BK cannot be cut by AL in any point M follows absolutely from Euclid I. 17, since otherwise in the same triangle MKL we would have two right angles at the points K and L, apart from the fact that in this case we would have our assertion about that angle BAN, that it is not in such circumstances the least of all.

But again AL cannot be the continuation of AN; because otherwise in the quadrilateral $NDKL$ we would have four right angles, against the hypothesis of acute angle.

But neither can it cut DN produced in any exterior point H; because angle AHN (from Euclid I. 16) would be acute, on account of the external angle AND supposed right; and therefore angle DHL would be obtuse, and so in the quadrilateral $DHLK$ we would have four angles, which taken together would be greater than four right angles, against the aforesaid hypothesis of acute angle.

Therefore it follows that the angle BAN must be cut by this AL, and therefore cannot be declared the least of all, drawn under which AN has with BX in two distinct points a common perpendicular ND.

Quod erat secundo loco demonstrandum. Itaque constat etc.

Corollary. But hence is permitted to observe, that under a lesser angle BAL is obtained (in hypothesis of acute angle) a common perpendicular LK, more remote indeed from the base AB, as follows from the construction, but moreover less than the other nearer common perpendicular ND, which is obtained under a greater angle BAN.

The reason of this latter is because in the quadrilateral $LKDS$ the angle at the point S is acute in the aforesaid hypothesis, since the three remaining angles are supposed right.

Wherefore (from Corollary I. to Proposition III.) the side LK will be less than the opposite side SD, and so much less than the side ND.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

III.

Construction of a One Parameter Group from an Infinitesimal Transformation.

9. Let there be given the one parameter continuous group

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a)\,; \tag{1}$$